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Speaking of Math

Marge Scherer

What is 3 + 2? When mathematics educator Liping Ma asked U.S. teachers that question, she was not seeking the answer 5.

"I stumbled upon the question by accident," she explains (2001). One day when she was thinking about how to make the teaching and learning of word problems more meaningful, she noticed that an important term was missing in English-language math vocabulary. In China, the concept is called *li shi*. Stating the *li shi* is the key step to solving a word problem. For example, for the word problem—John made three paper airplanes and Mike made two. How many did they make in all?—the *li shi* is 3 + 2.

In China, if the student is able to come up with a correct *li shi*, that student receives partial credit even if he or she does not compute the answer correctly.

When Ma asked, What do you call 3 + 2? educators offered responses ranging from *number sentence* to *horizontal problem*, but no answer got at her meaning. A textbook published in 1896 came closest, identifying 3 + 2 as a *mathematical expression*. For example, a sample question read, How many bushels of rye at 88 cents per bushel must be given for 33 hogsheads of molasses, each containing 63 gallons, at 80 cents per gallon? Students were then asked to write the mathematical expression. Interestingly, composing a mathematical expression is a basic skill taught in many elementary schools globally but rarely in the United States these days.

Why is this so important? Liping Ma explains that thinking about the mathematical expression of an idea focuses students less on coming up with correct answers and more on understanding mathematics. She writes:

In American elementary mathematics education, arithmetic is viewed as negligible. Many people seem to believe that arithmetic is only composed of a multitude of "math facts" and a handful of algorithms. . . . Who would expect the intellectual demand for learning such a subject actually is challenging and exciting?

How can we help students value mathematics for its intellectual challenge and exciting power to solve problems? Unfortunately, this question seems to have gone off the radar screen. According to a 2007 Public Agenda report called *Important, But Not for Me*, the majority of students and their parents polled believe that studying higher-level mathematics is not essential for life in the "real world." Students also said that they are most motivated to study higher-level math, not by the arguments about competing in the international economy, but by the need to fulfill college requirements. The poll did not even ask if they thought it important to learn math for the intellectual power it conveys.

Today's mathematical debates are complex and divisive. Among the questions of the day are, Should we require all students to learn higher mathematics? At what age should algebra be introduced? What can countries with vastly different cultures learn from one another? How can math be made relevant to today's students? Will subjects like trigonometry really be important to job-related skills of the future? And finally, what is the real purpose of learning mathematics?

Although our authors in this issue weigh in differently on some of the other questions, they do agree on the importance of making math more meaningful to students.

Marilyn Burns (p. 16) writes,

Only when the basics include understanding as well as skill proficiency will all students learn what they need for their continued success.

And Lynn Arthur Steen (p. 8) writes,

Unless teachers of all subjects—both academic and vocational—use mathematics regularly and significantly in their courses, students will treat mathematics teachers' exhortations about its usefulness as self-serving rhetoric. . . . Students in high school need much more practice using the mathematical resources introduced in the elementary and middle grades. Much of this practice should take place across the curriculum. Mathematics is too important to leave to mathematics teachers alone.

The question Liping Ma raises—How can we help students better understand math?—seems to count most of all.

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How Mathematics Counts

Lynn Arthur Steen

Fractions and algebra represent the most subtle, powerful, and mind-twisting elements of school mathematics. But how can we teach them so students understand?

Much to the surprise of those who care about such things, mathematics has become the 600-pound gorilla in U.S. schools. High-stakes testing has forced schools to push aside subjects like history, science, music, and art in a scramble to avoid the embarrassing consequences of not making "adequate yearly progress" in mathematics. Reverberations of the math wars of the 1990s roil parents and teachers as they seek firm footing in today's turbulent debates about mathematics education.

Much contention occurs near the ends of elementary and secondary education, where students encounter topics that many find difficult and some find incomprehensible. In earlier decades, schools simply left students in the latter category behind. Today, that option is neither politically nor legally acceptable. Two topics—fractions and algebra, especially Algebra II—are particularly troublesome. Many adults, including some teachers, live their entire lives flummoxed by problems requiring any but the simplest of fractions or algebraic formulas. It is easy to see why these topics are especially nettlesome in today's school environment. They are exemplars of why mathematics counts and why the subject is so controversial.

Confounded by Fractions

What is the approximate value, to the nearest whole number, of the sum 19/20 + 23/25? Given the choices of 1, 2, 42, or 45 on an international test, more than half of U.S. 8th graders chose 42 or 45. Those responses are akin to decoding and pronouncing the word *elephant* but having no idea what animal the word represents. These students had no idea that 19/20 is a number close to 1, as is 23/25.

Neither, it is likely, did their parents. Few adults understand fractions well enough to use them fluently. Because people avoid fractions in their own lives, some question why schools (and now entire states) should insist that all students know, for instance, how to add uncommon combinations like 2/7 + 9/13 or how to divide 1 3/4 by 2/3. When, skeptics ask, is the last time any typical adult encountered problems of this sort? Even mathematics teachers have a hard time imagining authentic problems that require these exotic calculations (Ma, 1999).

Moreover, many people cannot properly express in correct English the fractions and proportions that *do* commonly occur, for instance, in ordinary tables of data. A simple example illustrates this difficulty (Schield, 2002). Even though most people know that 20 percent means 1/5 of something, many cannot figure out what the something is when confronted with an actual example, such as the table in Figure 1. Although calculators can help the innumerate cope with such exotica as 2/7 + 9/13 and 1 3/4 ÷ 2/3, they are of no help to someone who has trouble reading tables and expressing those relationships in clear English.

Figure 1. The Challenge of Expressing Numerical Data in Ordinary Language

Percentage Who Are Runners			
	Nonsmoker	Smoker	Total
Female	50%	(20%)	40%
Male	25%	10%	20%
Total	37%	15%	30%

Source: From *Schield Statistical Literacy Inventory: Reading and Interpreting Tables and Graphs Involving Rates and Percentages*, by M. Schield, 2002. Minneapolis, MN: Augsburg College, W. M. Keck Statistical Literacy Project. Copyright 2002 by M. Schield. Available: http://web.augsburg.edu/~schield/MiloPapers/StatLitKnowledge2r.pdf. Reprinted with permission.

Which of the following correctly describes the 20% circled in the table above?

a. 20% of runners are female smokers.
b. 20% of females are runners who smoke.
c. 20% of female smokers are runners.
d. 20% of smokers are females who run.

These examples illustrate two very different aspects of mathematics that apply throughout the discipline. On the one hand is calculation; on the other, interpretation. The one reasons *with* numbers to produce an answer; the other reasons *about* numbers to produce understanding. Generally, school mathematics focuses on the former, natural and social sciences on the latter. For lots of reasons—psychological, pedagogical, logical, motivational—students will learn best when teachers combine these two approaches.

There may be good reasons that so many children and adults have difficulty with fractions. It turns out that even mathematicians cannot agree on a single proper definition. One camp argues that fractions are just names for certain points on the number line (Wu, 2005), whereas others say that it's better to think of them as multiples of basic unit fractions such as 1/3, 1/4, and 1/5 (Tucker, 2006). Textbooks for prospective elementary school teachers exhibit an even broader and more confusing array of approaches (McCrory, 2006).

Instead of beginning with formal definitions, when ordinary people speak of fractions they tend to emphasize contextual meaning. Fractions (like all numbers) are human constructs that arise in particular social and scientific contexts. They represent the magnitude of social problems (for example, the percentage of drug addiction in a given population); the strength of public opinion (for example, the percentage of the population that supports school vouchers); and the consequences of government policies (for example, the unemployment rate). Every number is the product of human activity and is selected to serve human purposes (Best, 2001, 2007).

Fractions, ratios, proportions, and other numbers convey quantity; words convey meaning. For mathematics to make sense to students as something other than a purely mental exercise, teachers need to focus on the interplay of numbers and words, especially on expressing quantitative relationships in meaningful sentences. For users of mathematics, calculation takes a backseat to meaning. And to make mathematics meaningful, the three *R*s must be well blended in each student's mind.

Algebra for All?

Conventional wisdom holds that in Thomas Friedman's metaphorically flat world, all students, no matter their talents or proclivities, should leave high school prepared for both college and high-tech work (American Diploma Project, 2004). This implies, for example, that all students should master Algebra II, a course originally designed as an elective for the mathematically inclined. Indeed, more than half of U.S. states now require Algebra II for almost all high school graduates (Zinth, 2006).

Advocates of algebra advance several arguments for this dramatic change in education policy:

- Workforce projections suggest a growing shortage of U.S. citizens having the kinds of technical skills that build on such courses as Algebra II (Committee on Science, Engineering, and Public Policy, 2007).
- Employment and education data show that Algebra II is a “threshold course” for high-paying jobs. In particular, five in six young people in the top quarter of the income distribution have completed Algebra II (Carnevale & Desrochers, 2003).
- Algebra II is a prerequisite for College Algebra, the mathematics course most commonly required for postsecondary degrees. Virtually all college students who have not taken Algebra II will need to take remedial mathematics.
- Students most likely to opt out of algebra when it is not required are those whose parents are least engaged in their children's education. The result is an education system that

magnifies inequities and perpetuates socioeconomic differences from one generation to the next (Haycock, 2007).

Skeptics of Algebra II requirements note that other areas of mathematics, such as data analysis, statistics, and probability, are in equally short supply among high school graduates and are generally more useful for employment and daily life. They point out that the historic association of Algebra II with economic success may say more about common causes (for example, family background and peer support) than about the usefulness of Algebra II skills. And they note that many students who complete Algebra II also wind up taking remedial mathematics in college.

Indeed, difficulties quickly surfaced as soon as schools tried to implement this new agenda for mathematics education. Shortly after standards, courses, and tests were developed to enforce a protocol of "Algebra II for all," it became clear that many schools were unable to achieve this goal. The reasons included, in varying degrees, inadequacies in preparation, funding, motivation, ability, and instructional quality. The result has been a proliferation of "fake" mathematics courses and lowered proficiency standards that enable districts and states to pay lip service to this goal without making the extraordinary investment of resources required to actually accomplish it (Noddings, 2007).

Several strands of evidence question the unarticulated assumption that additional instruction in algebra would necessarily yield increased learning. Although this may be true in some subjects, it is far less clear for subjects such as Algebra II that are beset by student indifference, teacher shortages, and unclear purpose. For many of the reasons given, enrollments in Algebra II have approximately doubled during the last two decades (National Center for Education Statistics [NCES], 2005a). Yet during that same period, college enrollments in remedial mathematics and mathematics scores on the 12th grade National Assessment of Educational Progress (NAEP) have hardly changed at all (NCES, 2005b; Lutzer, Maxwell, & Rodi, 2007). Something is clearly wrong.

Although we cannot conduct a randomized controlled study of school mathematics, with some students receiving a treatment and others a placebo, we can examine the effects of the current curriculum on those who go through it. Here we find more disturbing evidence:

- One in three students who enter 9th grade fails to graduate with his or her class, leaving the United States with the highest secondary school dropout rate among industrialized nations (Barton, 2005). Moreover, approximately half of all blacks, Hispanics, and American Indians fail to graduate with their class (Swanson, 2004). Although mathematics is not uniquely to blame for this shameful record, it is the academic subject that students most often fail.
- One in three students who enter college must remediate major parts of high school mathematics as a prerequisite to taking such courses as College Algebra or Elementary Statistics (Greene & Winters, 2005).
- In one study of student writing, one in three students at a highly selective college failed to use any quantitative reasoning when writing about subjects in which quantitative evidence should have played a central role (Lutsky, 2006).

- College students in the natural and social sciences consistently have trouble expressing in precise English the meaning of data presented in tables or graphs (Schield, 2006).

One explanation for these discouraging results is that the trajectory of school mathematics moves from the concrete and functional (for example, measuring and counting) in lower grades to the abstract and apparently nonfunctional (for example, factoring and simplifying) in high school. As many observers have noted ruefully, high school mathematics is the ultimate exercise in deferred gratification. Its payoff comes years later, and then only for the minority who struggle through it.

In the past, schools offered this abstract and ultimately powerful mainstream mathematics curriculum to approximately half their students—those headed for college—and little if anything worthwhile to the other half. The conviction that has emerged in the last two decades that all students should be offered useful and powerful mathematics is long overdue. However, it is not yet clear whether the best option for all is the historic algebra-based mainstream that is animated primarily by the power of increasing abstraction.

Mastering Mathematics

Fractions and algebra may be among the most difficult parts of school mathematics, but they are not the only areas to cause students trouble. Experience shows that many students fail to master important mathematical topics. What's missing from traditional instruction is sufficient emphasis on three important ingredients: communication, connections, and contexts.

Communication

Colleges expect students to communicate effectively with people from different backgrounds and with different expertise and to synthesize skills from multiple areas. Employers seek the same things. They emphasize that formal knowledge is not, by itself, sufficient to deal with today's challenges. Instead of looking primarily for technical skills, today's business leaders talk more about teamwork and adaptability. Interviewers examine candidates' ability to synthesize information, make sound assumptions, capitalize on ambiguity, and explain their reasoning. They seek graduates who can interpret data as well as calculate with it and who can communicate effectively about quantitative topics (Taylor, 2007).

To meet these demands of college and work, K–12 students need extensive practice expressing verbally the quantitative meanings of both problems and solutions. They need to be able to write fluently in complete sentences and coherent paragraphs; to explain the meaning of data, tables, graphs, and formulas; and to express the relationships among these different representations. For example, science students could use data on global warming to write a letter to the editor about carbon taxes; civics students could use data from a recent election to write op-ed columns advocating for or against an alternative voting system; economics students could examine tables of data concerning the national debt and write letters to their representatives about limiting the debt being transferred to the next generation.

We used to believe that if mathematics teachers taught students how to calculate and English teachers taught students how to write, then students would naturally blend these skills to write clearly about quantitative ideas. Data and years of frustrating experience show just how naïve this belief is. If we want students to be able to communicate mathematically, we need to ensure that they both practice this skill in mathematics class and regularly use quantitative arguments in subjects where writing is taught and critiqued.

Connections

One reason that students think mathematics is useless is that the only people they see who use it are mathematics teachers. Unless teachers of all subjects—both academic and vocational—use mathematics regularly and significantly in their courses, students will treat mathematics teachers' exhortations about its usefulness as self-serving rhetoric.

To make mathematics count in the eyes of students, schools need to make mathematics pervasive, as writing now is. This can best be done by cross-disciplinary planning built on a commitment from teachers and administrators to make the goal of numeracy as important as literacy. Virtually every subject taught in school is amenable to some use of quantitative or logical arguments that tie evidence to conclusions. Measurement and calculation are part of all vocational subjects; tables, data, and graphs abound in the social and natural sciences; business requires financial mathematics; equations are common in economics and chemistry; logical inference is fundamental to history and civics. If each content-area teacher identifies just a few units where quantitative thinking can enhance understanding, students will get the message.

The example of many otherwise well-prepared college students refraining from using even simple quantitative reasoning to buttress their arguments shows that students in high school need much more practice using the mathematical resources introduced in the elementary and middle grades. Much of this practice should take place across the curriculum. Mathematics is too important to leave to mathematics teachers alone.

Contexts

One of the common criticisms of school mathematics is that it focuses too narrowly on procedures (algorithms) at the expense of understanding. This is a special problem in relation to fractions and algebra because both represent a level of abstraction that is significantly higher than simple integer arithmetic. Without reliable contexts to anchor meaning, many students see only a meaningless cloud of abstract symbols.

As the level of abstraction increases, algorithms proliferate and their links to meaning fade. Why do you invert and multiply? Why is $(a + b)^2 \neq a^2 + b^2$? The reasons are obvious if you understand what the symbols mean, but they are mysterious if you do not. Understandably, this apparent disjuncture of procedures from meaning leaves many students thoroughly confused. The recent increase in standardized testing has aggravated this problem because even those teachers who want to avoid this trap find that they cannot. So long as procedures predominate on high-stakes tests, procedures will preoccupy both teachers and students.

There is, however, an alternative to meaningless abstraction. Most applications of mathematical reasoning in daily life and typical jobs involve sophisticated thinking with elementary skills (for example, arithmetic, percentages, ratios), whereas the mainstream of mathematics in high school (algebra, geometry, trigonometry) introduces students to increasingly abstract concepts that are then illustrated with oversimplified template exercises (for example, trains meeting in the night). By enriching this diet of simple abstract problems with sophisticated realistic problems that require only simple skills, teachers can help students see that mathematics is really helpful for understanding things they care about (Steen, 2001). Global warming, college tuition, and gas prices are examples of data-rich topics that interest students but that can also challenge them with surprising complications. Such a focus can also help combat student boredom, a primary cause of dropping out of school (Bridgeland, DiIulio, & Morison, 2006).

Most important, the pedagogical activity of connecting meaning to numbers needs to take place in authentic contexts, such as in history, geography, economics, or biology—wherever things are counted, measured, inferred, or analyzed. Contexts in which mathematical reasoning is used are best introduced in natural situations across the curriculum. Otherwise, despite mathematics teachers' best efforts, students will see mathematics as something that is useful only in mathematics class. The best way to make mathematics count in the eyes of students is for them to see their teachers using it widely in many different contexts.

My “Aha!” Moment

Douglas Hofstadter, Distinguished Professor of Cognitive Science, Indiana University, Bloomington.

I first realized the deep lure of mathematics when, at about age 3, I thought up the “great idea” of generalizing the concept of 2×2 to what seemed to me to be the inconceivably fancier concept of $3 \times 3 \times 3$. My inspiration was that since 2×2 uses the concept of two-ness *twice*, I wanted to use the concept of three-ness *thrice*! It wasn't finding out the actual value of this expression (27, obviously) that thrilled me—it was the idea of the fluid conceptual structures that I could play with in my imagination that turned me on to math at that early age.

Another “aha” moment came a few years later, when I noticed that $3^2 \times 5^2$ is equal to $(3 \times 5)^2$. Once again I was playing around with structures, not trying to prove anything. (I didn't even know that proofs existed!) It thrilled me to discover this pattern, which of course I verified for other values and found mystically exciting.

I believe that teachers should encourage playfulness with mathematical concepts and should encourage the discoveries of patterns of whatever sort. Any time a child recognizes an unexpected pattern, it may evoke a sense of wonder.

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Nine Ways to Catch Kids Up

Marilyn Burns

How do we help floundering students who lack basic math concepts?

Paul, a 4th grader, was struggling to learn multiplication. Paul's teacher was concerned that he typically worked very slowly in math and “didn't get much done.” I agreed to see whether I could figure out the nature of Paul's difficulty. Here's how our conversation began:

Marilyn: Can you tell me something you know about multiplication?

Paul: [*Thinks, then responds*] 6 × 8 is 48.

Marilyn: Do you know how much 6 × 9 is?

Paul: I don't know that one. I didn't learn it yet.

Marilyn: Can you figure it out some way?

Paul: [*Sits silently for a moment and then shakes his head*.]

Marilyn: How did you learn 6 × 8?

Paul: [*Brightens and grins*] It's easy—goin' fishing, got no bait, 6 × 8 is 48.

As I talked with Paul, I found out that multiplication was a mystery to him. Because of his weak foundation of understanding, he was falling behind his classmates, who were multiplying problems like 683 × 4. Before he could begin to tackle such problems, Paul needed to understand the concept of multiplication and how it connects to addition.

Paul wasn't the only student in this class who was floundering. Through talking with teachers and drawing on my own teaching experience, I've realized that in every class a handful of students are at serious risk of failure in mathematics and aren't being adequately served by the instruction offered. What should we do for such students?

Grappling with Interventions

My exchange with Paul reminded me of three issues that are essential to teaching mathematics:

- It's important to help students make connections among mathematical ideas so they do not see these ideas as disconnected facts. (Paul saw each multiplication fact as a separate piece of information to memorize.)
- It's important to build students' new understandings on the foundation of their prior learning. (Paul did not make use of what he knew about addition to figure products.)
- It's important to remember that students' correct answers, without accompanying explanations of how they reason, are not sufficient for judging mathematical understanding. (Paul's initial correct answer about the product of 6×8 masked his lack of deeper understanding.)

For many years, my professional focus has been on finding ways to more effectively teach arithmetic, the cornerstone of elementary mathematics. Along with teaching students basic numerical concepts and skills, instruction in number and operations prepares them for algebra. I've developed lessons that help students make sense of number and operations with attention to three important elements—computation, number sense, and problem solving. My intent has been to avoid the "yours is not to question why, just invert and multiply" approach and to create lessons that are accessible to all students and that teach skills in the context of deeper understanding. Of course, even well-planned lessons will require differentiated instruction, and much of the differentiation needed can happen within regular classroom instruction.

But students like Paul present a greater challenge. Many are already at least a year behind and lack the foundation of mathematical understanding on which to build new learning. They may have multiple misconceptions that hamper progress. They have experienced failure and lack confidence.

Such students not only demand more time and attention, but they also need supplemental instruction that differs from the regular program and is designed specifically for their success. I've recently shifted my professional focus to thinking about the kind of instruction we need to serve students like Paul. My colleagues and I have developed lessons that provide effective interventions for teaching number and operations to those far behind. We've grappled with how to provide instruction that is engaging, offers scaffolded instruction in bite-sized learning experiences, is paced for students' success, provides the practice students need to cement fragile understanding and skills, and bolsters students' mathematical foundations along with their confidence.

In developing intervention instruction, I have reaffirmed my longtime commitment to helping students learn facts and skills—the basics of arithmetic. But I've also reaffirmed that "the basics" of number and operations for all students, including those who struggle, must address all three aspects of numerical proficiency—computation, number sense, and problem solving. Only when the basics include understanding as well as skill proficiency will all students learn what they need for their continued success.

Essential Strategies

I have found the following nine strategies to be essential to successful intervention instruction for struggling math learners. Most of these strategies will need to be applied in a supplementary setting, but teachers can use some of them in large-group instruction.

1. Determine and Scaffold the Essential Mathematics Content

Determining the essential mathematics content is like peeling an onion—we must identify those concepts and skills we want students to learn and discard what is extraneous. Only then can teachers scaffold this content, organizing it into manageable chunks and sequencing these chunks for learning.

For Paul to multiply 683 × 4, for example, he needs a collection of certain skills. He must know the basic multiplication facts. He needs an understanding of place value that allows him to think about 683 as 600 + 80 + 3. He needs to be able to apply the distributive property to figure and then combine partial products. For this particular problem, he needs to be able to multiply 4 by 3 (one of the basic facts); 4 by 80 (or 8 × 10, a multiple of 10); 4 and by 600 (or 6 × 100, a multiple of a power of 10). To master multidigit multiplication, Paul must be able to combine these skills with ease. Thus, lesson planning must ensure that each skill is explicitly taught and practiced.

2. Pace Lessons Carefully

We've all seen the look in students' eyes when they get lost in math class. When it appears, ideally teachers should stop, deal with the confusion, and move on only when all students are ready. Yet curriculum demands keep teachers pressing forward, even when some students lag behind. Students who struggle typically need more time to grapple with new ideas and practice new skills in order to internalize them. Many of these students need to unlearn before they relearn.

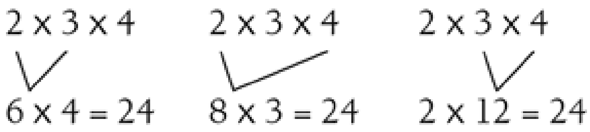
3. Build in a Routine of Support

Students are quick to reveal when a lesson hasn't been scaffolded sufficiently or paced slowly enough: As soon as you give an assignment, hands shoot up for help. Avoid this scenario by building in a routine of support to reinforce concepts and skills before students are expected to complete independent work. I have found a four-stage process helpful for supporting students.

In the first stage, the teacher models what students are expected to learn and records the appropriate mathematical representation on the board. For example, to simultaneously give students practice multiplying and experience applying the associative and commutative properties, we present them with problems that involve multiplying three one-digit factors. An appropriate first problem is 2 × 3 × 4. The teacher thinks aloud to demonstrate three ways of working this problem. He or she might say,

I could start by multiplying 2 × 3 to get 6, and then multiply 6 × 4 to get 24. Or I could first multiply 2 × 4, and then multiply 8 × 3, which gives 24 again. Or I could do 3 × 4, and then 12 × 2. All three ways produce the same product of 24.

As the teacher describes these operations, he or she could write on the board:



It's important to point out that solving a problem in more than one way is a good strategy for checking your answer.

In the second stage, the teacher models again with a similar problem—such as $2 \times 4 \times 5$—but this time elicits responses from students. For example, the teacher might ask, “Which two factors might you multiply first? What is the product of those two factors? What should we multiply next? What is another way to start?” Asking such questions allows the teacher to reinforce correct mathematical vocabulary. As students respond, the teacher again records different ways to solve the problem on the board.

During the third stage, the teacher presents a similar problem—for example, $2 \times 3 \times 5$. After taking a moment to think on their own, students work in pairs to solve the problem in three different ways, recording their work. As students report back to the class, the teacher writes on the board and discusses their problem-solving choices with the group.

In the fourth stage, students work independently, referring to the work recorded on the board if needed. This routine both sets an expectation for student involvement and gives learners the direction and support they need to be successful.

4. Foster Student Interaction

We know something best once we've taught it. Teaching entails communicating ideas coherently, which requires the one teaching to formulate, reflect on, and clarify those ideas—all processes that support learning. Giving students opportunities to voice their ideas and explain them to others helps extend and cement their learning.

Thus, to strengthen the math understandings of students who lag behind, make student interaction an integral part of instruction. You might implement the *think-pair-share* strategy, also called *turn and talk*. Students are first asked to collect their thoughts on their own, and then talk with a partner; finally, students share their ideas with the whole group. Maximizing students' opportunities to express their math knowledge verbally is particularly valuable for students who are developing English language skills.

5. Make Connections Explicit

Students who need intervention instruction typically fail to look for relationships or make connections among mathematical ideas on their own. They need help building new learning on what they already know. For example, Paul needed explicit instruction to understand how thinking about 6×8 could give him access to the solution for 6×9. He needed to connect the meaning of multiplication to what he already knew about addition (that 6×8 can be thought of as combining 6 groups of 8). He needed time and practice to cement this understanding for all multiplication problems. He would benefit from investigating six groups of other numbers—6 × 2, 6 × 3, and so on—and looking at the numerical pattern of these products. Teachers need to provide many experiences like these, carefully sequenced and paced, to prepare students like Paul to grasp ideas like how 6 × 9 connects to 6 × 8.

6. Encourage Mental Calculations

Calculating mentally builds students' ability to reason and fosters their number sense. Once students have a foundational understanding of multiplication, it's key for them to learn the basic multiplication facts—but their experience with multiplying mentally should expand beyond these basics. For example, students should investigate patterns that help them mentally multiply any number by a power of 10. I am concerned when I see a student multiply 18 × 10, for example, by reaching for a pencil and writing:

$$\begin{array}{r} 18 \\ \times\, 10 \\ \hline 00 \\ 18 \\ \hline 180 \end{array}$$

Revisiting students' prior work with multiplying three factors can help develop their skills with multiplying mentally. Helping students judge which way is most efficient to multiply three factors, depending on the numbers at hand, deepens their understanding. For example, to multiply 2 × 9 × 5, students have the following options:

$2 \times 9 \times 5$	$2 \times 9 \times 5$	$2 \times 9 \times 5$
$18 \times 5 = 90$	$10 \times 9 = 90$	$2 \times 45 = 90$

Guiding students to check for factors that produce a product of 10 helps build the tools they need to reason mathematically.

When students calculate mentally, they can estimate before they solve problems so that they can judge whether the answer they arrive at makes sense. For example, to estimate the product of 683 × 4, students could figure out the answer to 700 × 4. You can help students multiply 700 × 4 mentally by building on their prior experience changing three-factor problems to two-factor problems: Now they can change a two-factor problem—700 × 4—into a three-factor problem that includes a power of 10—7 × 100 × 4. Encourage students to multiply by the power of 10 last for easiest computing.

7. Help Students Use Written Calculations to Track Thinking

Students should be able to multiply 700 × 4 in their heads, but they'll need pencil and paper to multiply 683 × 4. As students learn and practice procedures for calculating, their calculating with paper and pencil should be clearly rooted in an understanding of math concepts. Help students see paper and pencil as a tool for keeping track of how they think. For example, to multiply 14 × 6 in their heads, students can first multiply 10 × 6 to get 60, then 4 × 6 to get 24, and then combine the two partial products, 60 and 24. To keep track of the partial products, they might write:

14 × 6

10 × 6 = 60

4 × 6 = 24

60 + 24 = 84

They can also reason and calculate this way for problems that involve multiplying by three-digit numbers, like 683 × 4.

8. Provide Practice

Struggling math students typically need a great deal of practice. It's essential that practice be directly connected to students' immediate learning experiences. Choose practice problems that support the elements of your scaffolded instruction, always promoting understanding as well as skills. I recommend giving assignments through the four-stage support routine, allowing for a gradual release to independent work.

Games can be another effective way to stimulate student practice. For example, a game like *Pathways* (see Figure 1 for a sample game board and instructions) gives students practice with multiplication. Students hone multiplication skills by marking boxes on the board that share a common side and that each contain a product of two designated factors.

Figure 1. Pathways Multiplication Game

Player 1 chooses two numbers from those listed (in the game shown here, 6 and 11) and circles the product of those two numbers on the board with his or her color of marker.

Player 2 changes just one of the numbers to another from the list (for example, changing 6 to 9, so the factors are now 9 and 11) and circles the product with a second color.

Player 1 might now change the 11 to another 9 and circle 81 on the board. Play continues until one player has completed a continuous pathway from one side to the other by circling boxes that share a common side or corner. To support intervention students, have pairs play against pairs.

72	36	49	88	54
84	77	96	132	56
63	81	48	108	121
(66)	99	144	64	42

6 7 8 9 **11** 12

9. Build In Vocabulary Instruction

The meanings of words in math—for example, *even, odd, product*, and *factor*—often differ from their use in common language. Many students needing math intervention have weak mathematical vocabularies. It's key that students develop a firm understanding of mathematical concepts before learning new vocabulary, so that they can anchor terminology in their understanding. We should explicitly teach vocabulary in the context of a learning activity and then use it consistently. A math vocabulary chart can help keep both teacher and students focused on the importance of accurately using math terms.

When Should We Offer Intervention?

There is no one answer to when teachers should provide intervention instruction on a topic a particular student is struggling with. Three different timing scenarios suggest themselves, each with pluses and caveats.

While the Class Is Studying the Topic

Extra help for struggling learners must be more than additional practice on the topic the class is working on. We must also provide comprehensive instruction geared to repairing the student's shaky foundation of understanding.

- *The plus*: Intervening at this time may give students the support they need to keep up with the class.
- *The caveat*: Students may have a serious lack of background that requires reaching back to mathematical concepts taught in previous years. The focus should be on the underlying math, not on class assignments. For example, while others are learning multidigit multiplication, floundering students may need experiences to help them learn basic underlying concepts, such as that 5 × 9 can be interpreted as five groups of nine.

Before the Class Studies the Topic

Suppose the class is studying multiplication but will begin a unit on fractions within a month, first by cutting out individual fraction kits. It would be extremely effective for at-risk students to have the fraction kit experience before the others, and then to experience it again with the class.

- *The plus*: We prepare students so they can learn *with* their classmates.
- *The caveat*: With this approach, struggling students are studying two different and unrelated mathematics topics at the same time.

After the Class Has Studied the Topic.

This approach offers learners a repeat experience, such as during summer school, with a math area that initially challenged them.

- *The plus*: Students get a fresh start in a new situation.
- *The caveat*: Waiting until after the rest of the class has studied a topic to intervene can compound a student's confusion and failure during regular class instruction.

How My Teaching Has Changed

Developing intervention lessons for at-risk students has not only been an all-consuming professional focus for me in recent years, but has also reinforced my belief that instruction—for all students and especially for at-risk students—must emphasize understanding, sense making, and skills.

Thinking about how to serve students like Paul has contributed to changing my instructional practice. I am now much more intentional about creating and teaching lessons that help intervention students catch up and keep up, particularly scaffolding the mathematical content to introduce concepts and skills through a routine of support. Such careful scaffolding may not be necessary for students who learn mathematics easily, who know to look for connections, and who have mathematical intuition. But it is crucial for students at risk of failure who can't repair their math foundations on their own.

My "Aha!" Moment

Mary M. Lindquist, Professor of Mathematics Education, Columbus College, Georgia. Winner of the National Council of Teachers of Mathematics Lifetime Achievement Award.

My "aha" moment came long after I had finished a masters in mathematics, taught mathematics in secondary school and college, and completed a doctorate in mathematics education. Although I enjoyed the rigor of learning and applying rules, mathematics was more like a puzzle than an elegant body of knowledge.

Many years of work on a mathematics program for elementary schools led to that moment. I realized that mathematics was more than rules—even the beginnings of mathematics were interesting. Working with elementary students and teachers, I saw that students could make sense of basic mathematical concepts and procedures, and teachers could help them do so. The teachers also posed problems to move students forward, gently let them struggle, and valued their approaches. What a contrast to how I had taught and learned mathematics!

With vivid memories of a number-theory course in which I memorized the proofs to 40 theorems for the final exam, I cautiously began teaching a number-theory course for prospective middle school teachers. My aha moment with these students was a semester long. We investigated number-theory ideas, I made sense of what I had memorized, and my students learned along with me. My teaching was changed forever.

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What's Right About Looking at What's Wrong?

Deborah Schifter

Both students and teachers gain new mathematical understanding by examining the reasoning behind a student's incorrect answer.

To teach mathematics for conceptual understanding, we need to treat it primarily as a realm of ideas to be investigated rather than a set of facts, procedures, and definitions to be used. To implement the former approach, teachers must have a deep understanding of content as well as the skill to implement concept-based pedagogy. And these greater demands on teachers, in turn, require well-thought-out forms of professional development. The following classroom lesson illustrates some of the issues involved.

Going Beyond Procedures

Liz Sweeney's 5th grade students all knew the standard procedure for multiplying multidigit numbers. On the day when a research team from the Education Development Center videotaped her class,[1] however, Ms. Sweeney wanted her students to go beyond the procedure. She asked them to find at least two ways to determine the products of several multidigit multiplication problems.

The students worked on this challenge, meeting in small groups to talk about their strategies. With just a few minutes left at the end of the period to discuss their work as a whole group, Ms. Sweeney asked Thomas to write his strategy for solving one of the problems (36 × 17) on the board, even though it was incorrect. Thomas wrote

Even Thomas knew his answer was wrong. Other strategies had already determined that the answer was 612. But he explained his reasoning to the class: To make the problem easier, he rounded up by adding 4 to 36 and 3 to 17; then he multiplied 40 × 20 to get 800, and subtracted the 4 and the 3 that he had added earlier, getting a final answer of 793.

Ms. Sweeney told the class what she had noticed as Thomas presented this method to his small group:

So I liked this—I was feeling comfortable with it, and it looked like a good strategy, and it was neat. And then Dima was all antsy in his seat, saying, "That's not what I did and my answer is really different". . .

So, tonight for your homework, I want you to copy down Thomas's method in your homework books, and I want you to figure out, What was Thomas thinking? And using the first steps of his strategy, how would you revise his approach to come up with a different answer?

Ms. Sweeney's behavior may puzzle readers whose images of effective teaching derive from the mathematics classrooms of their childhood. For many decades, mathematics has been taught the same way: The teacher demonstrates procedures for getting correct answers and then monitors students as they practice those procedures on a set of similar problems. Why did Ms. Sweeney ask her students, who already knew one efficient way to multiply 36 × 17, to find alternative strategies to do it? Why, at the end of class, did she ask a student to present a strategy that produced an incorrect result? And why did she ask the rest of the class to examine his strategy for homework?

When we view Ms. Sweeney's behavior from an alternative perspective, it becomes comprehensible. She acted on the belief that mathematics is much more than a set of discrete facts, definitions, and procedures to memorize and recall on demand. In her view, mathematics is an interconnected body of ideas to explore. To do mathematics is to test, debate, and revise or replace those ideas. Thus, the work of her class went beyond merely finding the answer to 36 × 17; it became an investigation of mathematical relationships.

Where Did Thomas's Error Come From?

This was not the first time Liz Sweeney had asked her students to think about different strategies for calculation. She had been assigning similar exercises for all four of the basic operations. By considering the *action* of the operation, students could develop such strategies independently. For example, when asked to add 18 + 24, students might consider the action of addition as the joining of two sets and devise a variety of methods for decomposing and recombining the addends:

- Decompose 18 into 10 and 8; decompose 24 into 20 and 4; add the tens, 10 + 20 = 30; add the ones, 8 + 4 = 12; add the results, 30 + 12 = 42.
- Take 2 from the 24 and add it to the 18. This becomes 20 + 22, or 42.
- Add 2 to 18 to get 20, 20 + 24 = 44. Then remove the 2 you have added on, 44 - 2 = 42.

The activity of devising calculation strategies and explaining why they work helps students cultivate several important mathematical capacities. Students develop a stronger number sense and become more fluent with calculation. They gain an understanding of place value when they decompose numbers into tens and ones. And they come to expect that mathematics will make sense and that they can solve problems through reasoning.

When Ms. Sweeney asked the class to multiply 36 and 17, Thomas decided to try out a strategy that he had used successfully to *add* two multidigit numbers: round up, perform the operation, and then subtract what had been added when rounding up. Thomas was reasoning by analogy, which is often a fruitful way to approach a problem. In this case, the analogy would not hold. But Thomas *was* reasoning; he was not merely careless.

Thomas's mistake—applying an addition strategy to a multiplication problem—is quite common. When faced with multidigit multiplication, such as 12 × 18, both children and adults frequently try (10 × 10) + (2 × 8). After all, to add 12 and 18, one could operate on the tens, operate on the ones, and then add the total. But multiplication involves a different kind of action, and thus requires a different set of adjustments after the factors have been changed or decomposed.

A Context for Multiplication

To think about the action of multiplication, it is helpful to envision a context in which the calculation might be used. For example, Thomas's classmate James thought of 36 × 17 as 36 bowls, each holding 17 cotton balls. With this context in mind, he could imagine an arrangement of bowls of cotton balls that would lend themselves to calculation.

James explained that first he arranged the bowls into groups of 10. Each group of 10 had 170 cotton balls (10 × 17), and there were three groups of ten (170 + 170 + 170). Besides the groups of 10 bowls, there were another 6 bowls with 17 cotton balls in each (6 × 17). To simplify that calculation, James thought of each bowl as having 10 white and 7 gray cotton balls, which yielded 60 white balls (6 × 10) plus 42 gray balls (6 × 7), for a total of 102 cotton balls in those 6 bowls. Then he added 170 + 170 + 170 + 102, which came out to 612.

A basic mathematical principle underlying James's method is the distributive property of multiplication over addition, which says that $(10 + 10 + 10 + 6) \times 17 = (10 \times 17) + (10 \times 17) + (10 \times 17) + (6 \times 17)$. The distributive property also says that $6 \times (10 + 7) = (6 \times 10) + (6 \times 7)$. James knew how to apply the distributive property, but when he worked with an image of cotton balls arranged in bowls, he was not merely manipulating numbers based on a set of rules he had memorized. He was able to perform the calculation as it made sense to him—that is, as it followed from his image of the context.

As Thomas, James, and their classmates developed their strategies in small groups, Ms. Sweeney went from group to group, sometimes asking questions or making suggestions and sometimes just listening. Having observed Thomas's mistaken strategy, she decided that it provided a learning opportunity for the class. When she gave the homework assignment, she was asking her students to go beyond evaluating whether the strategy was correct or not; she was challenging them to determine where it went wrong and how to make it right. To answer that question, students needed to examine closely the difference between addition and multiplication, highlighting the importance of thinking in terms of images like James's. This task also gave them an opportunity to state the distributive property explicitly. This one homework assignment yielded two further days of deep mathematical discussion in Ms. Sweeney's 5th grade class.

Teachers Consider Thomas's Strategy

In a professional development seminar,[2] my colleagues and I explored Ms. Sweeney's approach with a group of teachers. After viewing the video clip, many of the teachers were initially shocked by Ms. Sweeney's behavior. They didn't understand why she would “embarrass a student” by asking him to share his incorrect work. Some were dismayed that she would “punish the class” by assigning homework because one student made an error.

Rather than discuss these issues immediately, the facilitator asked the teachers to examine Thomas's strategy for themselves. After Thomas added 4 to 36 and 3 to 17, what would he need to subtract in order to get the correct result?

The teachers went to work in pairs and threes to examine different ways to approach the problem. The facilitator moved from group to group, listening to teachers, asking them to explain in more detail, and sometimes suggesting an approach. When each group had developed at least one way to think about the problem, the facilitator brought them all together to present their ideas.

Annie volunteered to share her initial thinking, which she realized was not completely correct. She said, “I did something that seems like it should be right, even though I know it's not.” She explained that when Thomas added 4 to 36 and 3 to 17 and then multiplied 40×20, he wasn't adding 4 units and 3 units, but 4 groups of units and 3 groups of units. She continued, So I first thought you need to subtract 4 groups of 17 and 3 groups of 36. But when I did the calculation, 800 - (4 × 17) - (3 × 36), I got 624—not 612, which we already know is the answer.

I didn't take away enough, so I thought maybe I multiplied by the wrong size group. Maybe I need to take away 4 groups of 20 and 3 groups of 40. But when I did this calculation, 800 - (4 × 20) - (3 × 40) = 600, I ended up with an answer that was too small!

I thought that was really strange. Then the facilitator came and suggested that we think of a story context.

A story context would allow the teachers to picture the steps of the problem, as James had done. Ming suggested the following context:

There are 40 children in a class, and they each paid $20 for a field trip. The teacher collected 40 × 20, or $800. But on the day of the field trip, 4 students were absent. That means she needed to give back $20 to each of those children, 800 - (4 × 20). Then the teacher went to the museum with 36 children, but when they got there they realized that the entrance fee was $17 instead of $20. That meant that each of the remaining 36 children got $3 back. So now we have 800 - (4 × 20) - (36 × 3), which the teacher paid to the museum. And that's $612—$17 for each of 36 children, or 36 × 17.

Ming added, “If you think about what Thomas did, it's like he gave each of the 4 absent students only $1, and he gave only 1 other student $3.”

Chad offered his group's use of an array, or the area of a rectangle, to think through the problem (see fig. 1). He explained,

The white part of the figures shows 36 × 17, and the gray regions show what gets added on when you change the problem to 40 × 20. In the picture on the right, you can see where Thomas went wrong. Instead of subtracting everything that got added on, he just took away what's shown in black.

You can see Ming's story in the diagram on the left. The gray region at the bottom stands for the money that was returned to the 4 children who were absent. The gray region on the right is the money that was returned to the 36 children who went on the field trip. The white region is the money that was paid to the museum.

Figure 1. Chad's Diagram

Annie pointed out that, when looking at Chad's diagram on the right, she can see more clearly why each of her initial answers was 12 off: “The first way I looked at it, I failed to subtract that little piece in the corner. The second way I looked at it, I subtracted that little piece twice.”

Aisha offered a fourth way of viewing the problem:

I wrote out the arithmetic and applied the distributive property: (36 + 4) × (17 + 3) = (36 × 17) + (36 × 3) + (4 × 17) + (4 × 3). So when Thomas multiplied 40 × 20, he needed to subtract those last three terms to get back to 36 × 17. When I was in high school, we called that procedure FOIL—you multiply the First terms, Outer terms, Inner terms, and Last terms. The thing is, I always did that because I was told that's the way to do it. But now that I can see it in the diagram, it really makes sense.

In this professional development session, participants offered four approaches to examine Thomas's strategy and figure out how to correct it. Note that, like Thomas, Annie chose to share her unresolved thinking. Looking together at what seems like it should be right, even though we know it's not, the teachers used several approaches to figure out where Annie's thinking went wrong. By sharing their different approaches, the teachers could compare approaches to see how one representation appeared in another.

The Professional Development Teachers Need

If teachers themselves were taught mathematics as discrete procedures and definitions to be memorized, how can schools prepare them to implement a more challenging, concept-based mathematics pedagogy? As a starting point, professional development needs to challenge teachers' conceptions of mathematics teaching and learning, opening them up to a process of reflection so that new insights can emerge.

Liz Sweeney's homework assignment provided just such an opportunity to the participants in the professional development seminar. Once the teachers had explored the mathematics in Thomas's error, they returned to their own questions about Sweeney's pedagogical approach. Among their comments were,

Of course all students know that addition and multiplication are different, but they don't always think about that. Our exploration of Thomas's error really highlights how you have to think about multiplication differently.

With these images, the distributive property isn't just a rule to memorize. You can see why it has to work.

I bet Thomas felt proud to have presented something that got his classmates thinking so hard.

Such insights cannot be induced by a series of lectures or workshops on instructional strategies. Instead, professional development programs need to dig deeper, giving their participants opportunities to construct more powerful understandings of learning, teaching, and disciplinary substance.

A first step in helping teachers change their pedagogy is to place them in seminars where they can explore disciplinary content, develop new conceptions of mathematics, and gain a heightened sense of their own mathematical powers. As learners of mathematics, they experience a new kind of classroom. In these seminars, teachers reflect on their own learning processes and consider those features of the classroom that support or hinder them. Through such professional development, we can inspire teachers to envision and implement a new kind of mathematics pedagogy—one in which student understanding and collaborative thinking take center stage.

My “Aha!” Moment

Jeremy Kilpatrick, Regents Professor of Mathematics Education, University of Georgia, Athens. Winner of the National Council of Teachers of Mathematics Lifetime Achievement Award.

Although I did well in mathematics in high school, it was not until I went to Chaffey College, a two-year college then located in Ontario, California, and took calculus from Arthur E. Flum, that I discovered that learning mathematics could be simultaneously difficult and enjoyable, elegant and fascinating. The moment I realized all this came during the first week of class, when Mr. Flum's infectious enthusiasm for the subject we were about to work on together became apparent. Calculus was a new world for us, but under his guidance, we would succeed not only in learning it but in seeing its power and elegance. I ended up taking every mathematics course I could from Mr. Flum, and when I transferred as a junior to Cal Berkeley, mathematics was the obvious subject in which to major.

When I learned later that research on effective teachers has repeatedly shown that enthusiasm is one of their signature traits, I thought of Mr. Flum. In all that he did—coaching the tennis team, sponsoring the booster club, teaching mathematics—he had a flair for pushing you harder while helping you enjoy what you were doing. Successful mathematics teachers are enthusiastic about mathematics, and that enthusiasm is contagious.

Endnotes

[1] This classroom episode can be seen in the video component of Schifter, D., Bastable, V., & Russell, S. J. (1999). *Building a system of tens*. Parsippany, NJ: Pearson.

[2] The session described here is a composite of several seminar groups that were part of the *Developing Mathematical Ideas* professional development program.

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